Review:

- Dynamical sys.

$$\dot{x} = f(x), \qquad x(0) = x_0$$

- $x \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^n$

- $\bar{x}$ is an eqlb point if $f(\bar{x}) = 0$

- Nonlinear phenomenons

Today

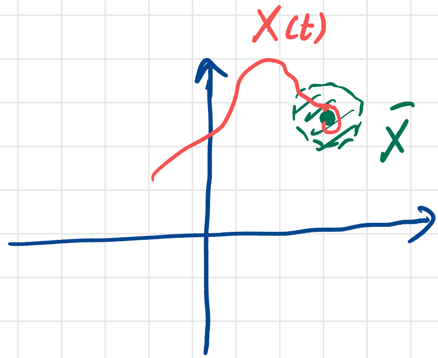
Ch2.1-2.3 (Khalil)

1. Linearization around eqlb. points

2. Classification of nodes

Linearization:

- Explain the behaviour of $x(t)$ around eqlb. $\bar{x}$ by a linear system



- Let's look at 2-dim systems (for simplicity)

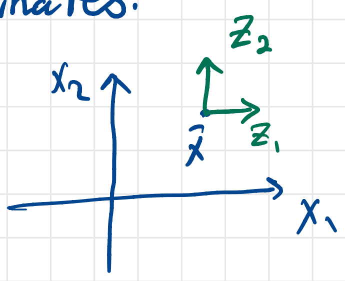
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

- let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be eqlb. point

- objective: obtain a linearized system around $\bar{x}$

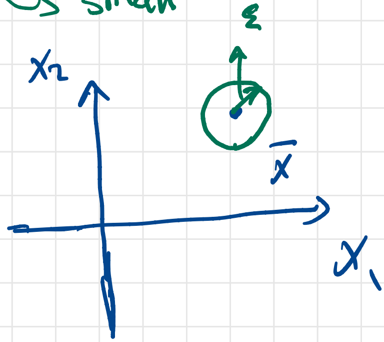
- step 1: change of coordinates:

$$z = x - \bar{x}$$



- Step 2: assume $\underbrace{\|z(t)\|}_{\text{Euclidean norm}} \leq \varepsilon \rightsquigarrow$ small

$\|z\| = \sqrt{z_1^2 + z_2^2}$



- Step 3: Taylor series expansion around $\bar{x}$

$$\dot{z}_1 = f_1(\bar{x}_1 + z_1, \bar{x}_2 + z_2)$$

$$= \underbrace{f_1(\bar{x}_1, \bar{x}_2)}_{0} + \frac{\partial f_1}{\partial x_1}(\bar{x}_1, \bar{x}_2) z_1 + \frac{\partial f_1}{\partial x_2}(\bar{x}_1, \bar{x}_2) z_2 + O(\varepsilon^2)$$

Similarly

$$\dot{z}_2 = f_2(\bar{x}_1 + z_1, \bar{x}_2 + z_2)$$

$$= \frac{\partial f_2}{\partial x_1}(\bar{x}_1, \bar{x}_2) z_1 + \frac{\partial f_2}{\partial x_2}(\bar{x}_1, \bar{x}_2) z_2 + O(\varepsilon^2)$$

- Ignoring $O(\varepsilon^2)$

$$\Rightarrow \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}) & \frac{\partial f_1}{\partial x_2}(\bar{x}) \\ \frac{\partial f_2}{\partial x_1}(\bar{x}) & \frac{\partial f_2}{\partial x_2}(\bar{x}) \end{bmatrix}}_{\text{Jacobian}} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}(\bar{x})$$

$\Longrightarrow$

$$\boxed{\begin{matrix} \dot{x} = f(x) \\ f(\bar{x}) = 0 \end{matrix} \xrightarrow[\text{around } \bar{x}]{\text{linearization}} \begin{matrix} z = x - \bar{x} \\ \dot{z} = Az, \quad A = \frac{\partial f}{\partial x}(\bar{x}) \end{matrix}}$$

- generalization to n-dim

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \dots, & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1}, & \dots & , & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

- Back to pandelum example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\omega^2 \sin x_1 - \gamma x_2 \end{bmatrix}$$

$$\Longrightarrow \frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1$$

$$\frac{\partial f_2}{\partial x_1} = -\omega^2 \cos(x_1), \quad \frac{\partial f_2}{\partial x_2} = -\gamma$$

$$\Longrightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos(x_1) & -\gamma \end{bmatrix}$$

- There are two eqlb. points.

$$\bar{x}^{(1)} = (0, 0), \quad \bar{x}^{(2)} = (\pi, 0)$$

- linearized sys. around $\bar{x}^{(1)}$

$$\dot{z} = Az, \quad A = \frac{\partial f}{\partial x}(\bar{x}^{(1)}) = \begin{bmatrix} 0 & 1 \\ -w^2 & -\gamma \end{bmatrix}$$

- linearized sys. around $\bar{x}^{(2)}$

$$\dot{z} = Az, \quad A = \frac{\partial f}{\partial x}(\bar{x}^{(2)}) = \begin{bmatrix} 0 & 1 \\ w^2 & -\gamma \end{bmatrix}$$

Classification of eqlb. points:

- The "qualitative" behavior of nonlinear sys. around eqlb. is determined by the linearized sys.
- The behavior of linear sys. $\hat{z} = Az$ depends on eigenvalues of A.
- Assume 2-dim sys.

$\Longrightarrow$ A has two eigen values; λ_1, λ_2

- We consider six cases

we assume λ_1, λ_2 are distinct and non-zero

① λ_1, λ_2 are real

- $\xrightarrow{1\text{-}a}$ $\lambda_1, \lambda_2 > 0$
- $\xrightarrow{1\text{-}b}$ $\lambda_1 > 0, \ \lambda_2 < 0$
- $\xrightarrow{1\text{-}c}$ $\lambda_1, \lambda_2 < 0$

② λ_1, λ_2 are complex

$\lambda_1 = \alpha + i\beta$

$\lambda_2 = \alpha - i\beta$

- $\xrightarrow{2\text{-}a}$ $\alpha > 0$
- $\xrightarrow{2\text{-}b}$ $\alpha = 0$
- $\xrightarrow{2\text{-}c}$ $\alpha < 0$

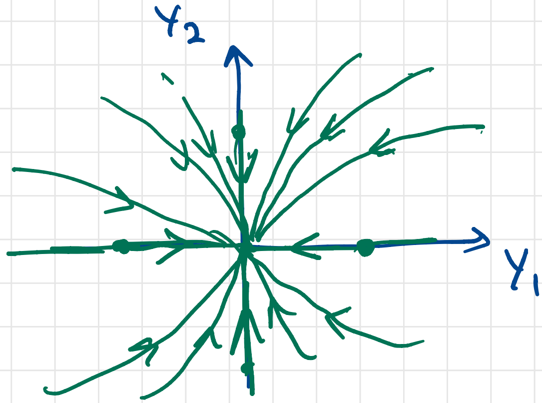
① λ_1, λ_2 are real

with a change of basis $\longrightarrow$ $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Solution in new basis $\quad y_1 = e^{\lambda_1 t} y_1(0)$
$\quad y_2 = e^{\lambda_2 t} y_2(0)$

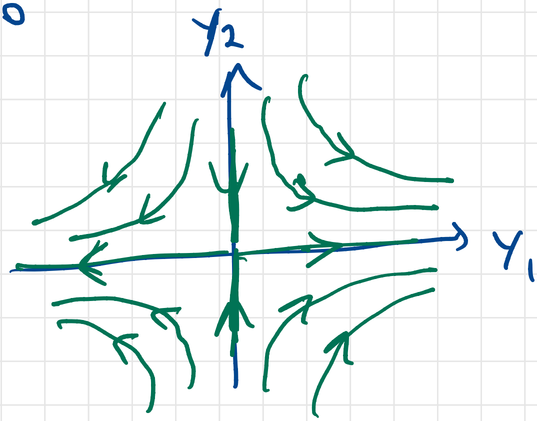
a) $\lambda_1 < \lambda_2 < 0$

stable node



b) $\lambda_1 > 0, \lambda_2 < 0$

saddle



c) $\lambda_1 > \lambda_2 > 0$

unstable node

same as (a)
but going outward.

② λ_1, λ_2 are complex

$\lambda_1 = \alpha + i\beta$

$\lambda_2 = \alpha - i\beta$

with change of basis $\rightarrow$ $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

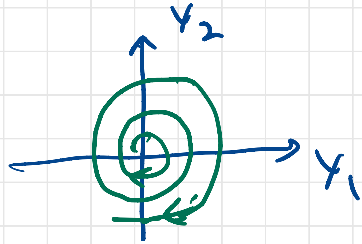
Solution in new basis

$Y_1 = e^{\alpha t} \cos \beta t$

$Y_2 = e^{\alpha t} \sin \beta t$

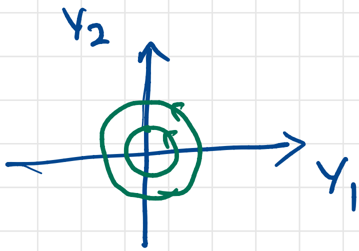
a) $\alpha > 0$

unstable focus



b) $\alpha = 0$

center



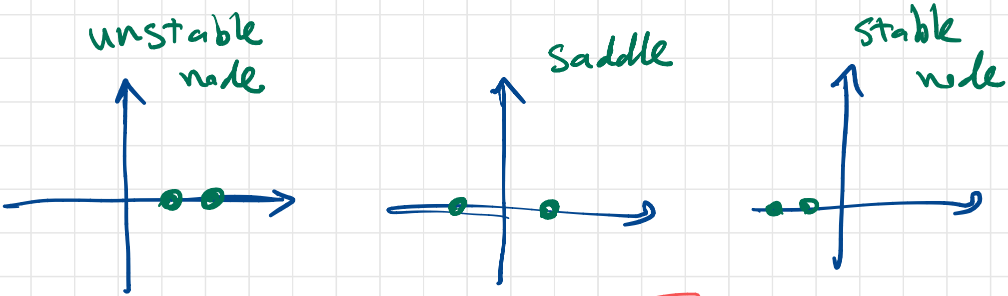
c) $\alpha < 0$

stable focus

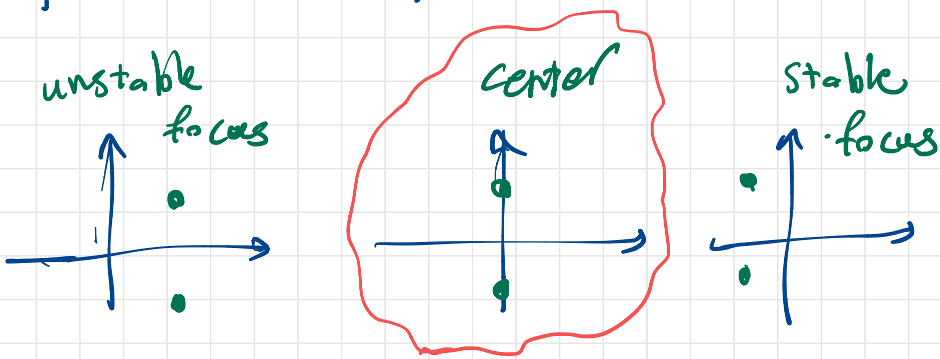
same as (a) but inward spiral

To summerize: location of eigenvalues in complex plane.

- a)



b)



- The linearized sys. explains qulatitative behaviour of nonlinear sys. around eqlb. except when it is center.

center is not robust against perturbation

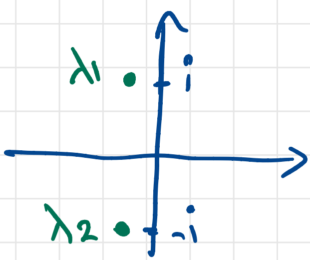
Example: Pendulum

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ +\omega^2 & -\gamma \end{bmatrix}$$

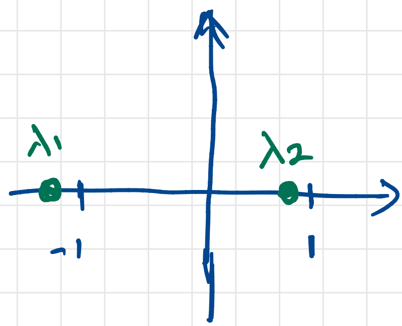
assume $\omega = 1$, $\gamma = 0.1$

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2} \qquad\qquad \lambda = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2}$$

$$\simeq -0.1 \pm i \qquad\qquad \simeq -1.1,\ 0.9$$



stable focus



saddle

see handout for phase-portrait

Example:

$$\dot{x}_1 = -x_2 - \mu x_1 (x_1^2 + x_2^2)$$

$$\dot{x}_2 = x_1 - \mu x_2 (x_1^2 + x_2^2)$$

$\bar{x} = 0$ is eqlb. point

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -3\mu x_1^2 - \mu x_2^2 , & -1 - 2\mu x_1 x_2 \\ 1 - 2\mu x_1 x_2 & , -3\mu x_2^2 - \mu x_1^2 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}(0) = \begin{bmatrix} 0 , & -1 \\ 1 , & 0 \end{bmatrix}$$

$\lambda_1 = +i$, $\lambda_2 = -i$ $\Rightarrow$ center

however the nonlinear sys is stable focus if $\mu > 0$

or unstable focus if $\mu < 0$

see handout for portrait

Generalization to n-dim sys.

Thm: (to be proved later)

let $\bar{x}$ be eqlb. point for $\dot{x} = f(x)$ and $A = \frac{\partial f}{\partial x}(\bar{x})$. Then

(a) if $Re(\lambda) < 0$ for all eigenvalues λ then $\bar{x}$ is "stable".

(b) if $Re(\lambda) > 0$ for some λ, then $\bar{x}$ is <u>not</u> stable

(c) if $Re(\lambda) \leq 0$ for all λ, but $Re(\lambda) = 0$ for some λ, then nothing can be said.